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Vacuum polarization and low-energy scattering

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Abstract. The effects of vacuum polarization on the S-wave scattering of protons by protons and alpha particles by alpha particles are investigated. The calculations are exact. The inclusion of vacuum polarization gives a slightly improved effective range fit to the p–p scattering data. The width of the ground state of beryllium is calculated from the α–α scattering data. The value of $6{\cdot}14 \pm 0{\cdot}04$ ev compares favourably with the experimental value of $6{\cdot}8 \pm 1{\cdot}7$ ev.

1. Introduction

Experiments for studying the elastic scattering of protons by protons and alpha particles by alpha particles have now been performed over a wide range of energies, and the measured differential cross sections analysed in terms of 'nuclear' phase shifts (Lauritsen and Ajzenberg-Selove 1966 (α–α experiments), Hulthén and Sugawara 1957, Noyes 1964 (p–p experiments)). Since both protons and alpha particles are charged, the 'nuclear' phase shift is the additional phase shift due to the 'nuclear' potential embedded in the Coulomb potential. The differential cross section for the scattering of similar particles in the centre-of-mass system is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)+sf(\pi-\theta)|^2$$

where the sign s is positive for alpha particles and negative for protons. The scattering amplitude $f(\theta)$ can be written

$$f(\theta) = f_{\mathrm{C}}(\theta)+f_{\mathrm{N}}(\theta)$$

where $f_{\mathrm{C}}(\theta)$ is the scattering amplitude for the Coulomb potential. It is given by

$$f_{\mathrm{C}}(\theta) = \frac{\eta}{2k}\sin^2\tfrac{1}{2}\theta \exp\{-i\eta \log(\sin^2\tfrac{1}{2}\theta)+2i\sigma_0\}$$

where

$$\sigma_L = \arg(L+1+i\eta)$$
$$\eta = \frac{Z_1Z_2e^2}{\hbar v} = \frac{b}{2k}$$

and k is the relative momentum of the two particles. $f_{\mathrm{N}}(\theta)$ is the additional scattering amplitude due to the 'nuclear' potential and can be written

$$f_{\mathrm{N}}(\theta) = (2ik)^{-1}\sum_L (2L+1)\exp(2i\sigma_L)\{\exp(2i\delta_L)-1\}\mathrm{P}_L(\cos\theta)$$

where δ_L is the 'nuclear' phase shift and $\mathrm{P}_L(\cos\theta)$ is the Legendre polynomial. For the proton case allowance must also be made for the intrinsic spin. The calculation of the phase shifts δ_L from the differential cross sections is by no means a trivial one, and often does not give a unique set. The details of this calculation are given in the literature (e.g. Tombrello and Senhouse 1963).

The variation of δ_L with energy shows quite clearly in both cases that the nuclear potential is repulsive at short distances. For p–p elastic scattering the singlet S-wave phase shift becomes negative at high incident energies (about 150 MeV in the centre-of-mass

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system) (Preston 1962, p. 102). If this repulsion is due to a hard core, as opposed to a strong but finite potential, the radius of the hard core is about 0·4 fm, although its precise value is a little uncertain. Using a phenomenological singlet potential, Hamada and Johnston (1962) obtained a hard core radius of 0·343 fm. In recent work by Heller *et al.* (1964) this value was modified to 0·48 fm (see also Noyes 1965). For α–α scattering the repulsion is more marked. In addition to the hard core in the nucleon–nucleon interaction, there is the quantal repulsion resulting from the fact that the constituent nucleons of the alpha particles obey Fermi statistics. The combined repulsion can be considered as arising from a hard core at a radius of about 1·7 fm. Its inclusion greatly facilitates the application of effective-range theory (Kermode 1967).

From the phase shift δ_L we can define the function

$$y_L(a_L) = k\left\{\frac{G_L'(\eta,\rho)+F_L'(\eta,\rho)\cot\delta_L}{G_L(\eta,\rho)+F_L(\eta,\rho)\cot\delta_L}\right\}_{a_L} \tag{1}$$

where $F_L(\eta,\rho)$ ($G_L(\eta,\rho)$) is the regular (irregular) Coulomb wave function, a_L is the hard core radius, $\rho = kr$, and the prime means differentiation with respect to ρ. If δ_L results from a *short-range* interaction, then $y_L(a_L)$ is a slowly varying function of energy and can be written (Kermode 1965, 1967)

$$y_L(a_L) = B_{L,0}+B_{L,1}k^2+B_{L,2}k^4. \tag{2}$$

Equation (2), the effective-range formula, also holds for a non-local interaction, provided that the kernel is symmetrical. In this paper we shall consider the case of S-wave scattering only. For no hard core, equation (2) reproduces the well-known formula of Jackson and Blatt (1950), where

$$y_0(0) = b\left\{\frac{\pi\cot\delta_0}{\exp(2\pi\eta)-1}+h(\eta)\right\} \tag{3}$$

and

$$h(\eta) = -0{\cdot}5772\ldots-\log\eta+\sum_{r=1}^{\infty}\frac{\eta^2}{r(r^2+\eta^2)}.$$

In § 2 we assume that equation (2) is valid, i.e. the interactions have a short range, and use the experimental phase shifts to calculate the coefficients $B_{0,i}$ ($i = 0, 1, 2$). In § 3 we introduce the vacuum polarization potential. This potential has a long range. For the remainder of the paper we investigate the effects arising from the inclusion of this potential in the scattering problem.

2. Effective-range analysis (i): short-range forces

The p–p data were first analysed under the assumption that only *short-range* forces were responsible for the experimentally determined 'nuclear' phase shifts δ_0. Since the phase shifts tabulated by Noyes (1964) were more accurate than those tabulated by Hulthén and Sugawara (1957), it was decided to analyse each set of data separately. The latter paper, on the other hand, contained a larger number of experimental results. A comparison of the two analyses shows the extent to which the quality and quantity of these experimental results balance.

Of the 35 experimental values for δ_0 given in the review article by Hulthén and Sugawara, five were rejected because of the uncertainty in energy. In addition, the experiment at 2·1 MeV (c.m.) was not considered because the phase shift δ_1 had not been included in the analysis of the differential cross section. The remaining 29 points were contained in the centre-of-mass energy region 0·1–4·865 MeV. From each of the 29 experimental phase shifts, y_0 (0·48) was calculated on the IBM 7040 computer at Liverpool University. The Coulomb wave functions were calculated by the series expansion method described by Froberg (1955). The errors in the experimental data were taken as given in the literature. This was in contrast with the analysis by Hulthén and Sugawara in which equal weight was assigned to all the experimental data. The coefficients $B_{0,i}$ ($i = 0, 1, 2$) were chosen

to minimize the function

$$S^2 = n^{-1} \sum_{i=1}^{n} \sigma_i^{-2}(y_{0_i} - B_{0,0} - B_{0,1}k_i^2 - B_{0,2}k_i^4)^2. \quad (4)$$

In equation (4), σ_i is the standard deviation associated with y_{0_i} and the summation extends to all the experimental points. The results of this analysis, p–p (1), are given in table 1.

A separate effective-range analysis was performed using the five phase shifts tabulated by Noyes (1964). The centre-of-mass energy range for this case extended from 0·191 25 MeV to 1·5185 MeV. The resulting coefficients, labelled p–p (2), are also given in table 1.

A similar analysis of the α–α S-wave data up to 12 MeV (c.m.) has already been reported (Kermode 1967). In this analysis the y_0 (1·7) curve was constrained to pass through the resonance energy E_r, the ground state of beryllium. The value of E_r has been given variously as $92{\cdot}12 \pm 0{\cdot}05$ kev (Benn *et al.* 1966) and $94{\cdot}5 \pm 0{\cdot}5$ kev (Lauritsen and Ajzenberg-Selove 1966). A separate analysis was performed for each of these values. The results are given in table 1.

Table 1. Coefficients $B_{0,i}$ obtained from the experimental phase shifts

Experiment	No. of points	a_0 (fm)	$B_{0,0}$ (fm^{-1})	$B_{0,1}$ (fm)	$B_{0,2}$ (fm^3)	S^2
p–p (1)	29	0·48	0·026845	0·89182	−1·1028	1·90
p–p (2)	5	0·48	0·026989	0·84374	0·27392	0·43
α–α (1)	37	1·7	−0·45586	0·79680	0·00924	0·42
α–α (2)	37	1·7	−0·45571	0·79569	0·01022	0·41

Vacuum polarization effects are neglected.

In this table the results for α–α scattering are α–α (1) or α–α (2), depending on whether we take the resonance at 92·12 kev or 94·5 kev. Using the coefficients given in table 1, the width of the ground state of beryllium was calculated from the formula (Kermode 1967)

$$\Gamma = \left\{ \frac{2k}{G_0^2(dz_0/dE - dy_0/dE)} \right\}_{E_r, a_0} \quad (5)$$

where $z_0 = kG_0'/G_0$. The width was found to be $6{\cdot}14 \pm 0{\cdot}04$ ev (for $E_r = 92{\cdot}12 \pm 0{\cdot}05$ kev) or $7{\cdot}7 \pm 0{\cdot}4$ ev (for $E_r = 94{\cdot}5 \pm 0{\cdot}5$ kev). The errors in these values arose mainly from the uncertainty in the position of the resonance. The effect of the errors in the coefficients $B_{0,i}$ was small ($\simeq 0{\cdot}01$ ev). The width of $6{\cdot}14 \pm 0{\cdot}04$ ev compares very favourably with the experimental value of $6{\cdot}8 \pm 1{\cdot}7$ ev (Benn *et al.* 1966).

Equation (5) is valid whether δ_0 is produced by short-range forces or not. The advantage of the short-range assumption is that equation (2) is also valid. This enables the gradient of $y_0(a_0)$ at the resonance to be calculated accurately from moderately accurate scattering data at higher energies. In the following we consider the effect of a particular long-range force, i.e. vacuum polarization, in addition to the Coulomb and short-range forces. For this case equation (2) is no longer valid and the calculation of dy_0/dE becomes more difficult.

3. Vacuum polarization

From the theory of quantum electrodynamics it is known that in addition to the Coulomb potential between charged particles there are other effects which, although small, may be significant for a description of the experimental results (Feynman 1961). The largest effect, arising from the diagram shown in figure 1, is that due to vacuum polarization. This interaction was first discussed by Uehling (1935) and was later found to contribute an important part of the total Lamb shift between the electronic energy levels 2s $^2S_{1/2}$ and 2p $^2P_{1/2}$ in the hydrogen atom (Lamb 1951, Triebwasser *et al.* 1953).

The effects of vacuum polarization in proton–proton scattering have been considered by Foldy and Eriksen (1954, 1955), Eriksen *et al.* (1956), Durand (1957) and Heller (1960).

The change in the function $K = b^{-1}y_0(0)$ (equation (3)) resulting from the vacuum polarization was calculated by Foldy and Eriksen (1954) using first-order perturbation theory. They found that the phase shifts were not sufficiently accurate to confirm the presence of vacuum polarization in S-wave scattering. For P waves, however, the calculations of Eriksen *et al.* and Durand showed that approximately half of the observed phase

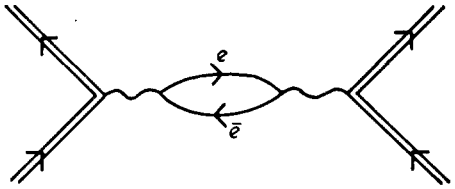

Figure 1. Feynman diagram for vacuum polarization.

shift at an energy in the range 2–5 MeV (lab.) was due to vacuum polarization. In the paper by Heller (1960) the effective-range formula for a short-range potential in a combined Coulomb *and* vacuum polarization potential was derived. The resulting formula is rather complicated.

Both of these approaches have been incorporated in the present work. There are, however, two differences in the calculations. First, it is assumed that there is a hard core in the nuclear potential, in a similar manner to the previous analysis without vacuum polarization. Secondly, the change in the function $y_0(a_0)$ is calculated exactly and not by first-order perturbation theory. Calculations have been made for both p–p and α–α scattering since these show different behaviour. The reason for this is that the nuclear phase shifts for p–p scattering are positive for small energies, whereas for α–α scattering they are negative for energies above the resonance (i.e. negative with respect to 180°). The phase shift due to the vacuum polarization potential is negative for both cases. Hence deviations from pure Mott scattering due to the nuclear potential are diminished in the proton case, but enhanced in the alpha case by the inclusion of the vacuum polarization effect.

Using the integral form of the Uehling potential given by Schwinger (1949), the combined Coulomb and vacuum polarization potential may be written

$$V_E(r) = \frac{b}{r}\{1+\lambda I(r)\} \quad \text{fm}^{-2} \tag{6}$$

where

$$I(r) = \int_1^\infty e^{-2\kappa r x}\left(\frac{1}{x^2}+\frac{1}{2x^4}\right)(x^2-1)^{1/2}\,dx \tag{7}$$

$$(2\kappa)^{-1} = \hbar/2mc = 193{\cdot}1 \quad \text{fm}$$

and

$$\lambda = 2\alpha/3\pi = 1{\cdot}549\times10^{-3}.$$

The limiting forms of $I(r)$ are

$$I(r) = -0{\cdot}5772\ldots -\tfrac{5}{6}-\log(\kappa r)+O(2\kappa r); \qquad 2\kappa r \ll 1$$

$$I(r) = \frac{3(2\pi)^{1/2}}{4}\frac{e^{-2\kappa r}}{(2\kappa r)^{3/2}}\left\{1+O\left(\frac{1}{2\kappa r}\right)\right\}; \qquad 2\kappa r \gg 1.$$

The integral (7) for a given r was evaluated numerically by the method outlined in the appendix. This method, which is extremely quick ($\simeq$ 50 ms on the IBM 7040), also gives the value of $dI(r)/dr$.

Following Heller, we call the combined Coulomb and vacuum polarization potential the electric potential and the solutions of the Schrödinger equation the electric wave functions. The regular and irregular electric wave functions, denoted by $S_L(r)$ and $T_L(r)$

respectively, have the asymptotic forms

$$\begin{aligned} S_L(r) &\underset{2\kappa r\gg 1}{\sim} F_L(r)+\tan\tau_L\, G_L(r) \\ T_L(r) &\underset{2\kappa r\gg 1}{\sim} G_L(r)-\tan\tau_L\, F_L(r) \end{aligned} \tag{8}$$

where τ_L is the additional phase shift due to vacuum polarization.

For nuclear scattering in the presence of the electric potential the effective-range formula (equation (2)) has to be modified on account of the long-range nature of vacuum polarization. The new formula, which can be derived in a similar manner to equation (2) (Kermode 1965), is

$$y_L{}^{\mathrm{E}}(a_L) = B_{L,0}{}^{\mathrm{E}}+B_{L,1}{}^{\mathrm{E}}k^2+B_{L,2}{}^{\mathrm{E}}k^4 \tag{9}$$

where

$$y_L{}^{\mathrm{E}}(a_L) = k\left(\frac{T_L'+S_L'\cot\delta_L{}^{\mathrm{E}}}{T_L+S_L\cot\delta_L{}^{\mathrm{E}}}\right)_{a_L} \tag{10}$$

and $\delta_L{}^{\mathrm{E}}$ is the nuclear phase shift due to the hard core and the nuclear potential embedded in the electric potential. From the asymptotic wave functions it follows that

$$\delta_L = \delta_L{}^{\mathrm{E}}+\tau_L. \tag{11}$$

If we consider S waves, taking the limit as a_0 tends to zero and performing any necessary subtraction of infinite terms (Kermode 1967), equation (9) should reproduce Heller's result.

4. Riccati equation

From the experimental data for p–p and α–α scattering, we wish to calculate the function $y_0{}^{\mathrm{E}}(a_0)$ and hence the coefficients $B_{0,i}{}^{\mathrm{E}}$ $(i = 0, 1, 2)$. For a direct calculation of $y_0{}^{\mathrm{E}}(a_0)$ one needs the vacuum polarization phase shift τ_0, the electric wave functions and their derivatives. Unlike the case of the Coulomb wave functions there is no known series expansion for the electric functions. Furthermore, the phase shifts given in the literature are the δ_0 and not the $\delta_0{}^{\mathrm{E}}$. To circumvent these difficulties the $y_0{}^{\mathrm{E}}(a_0)$ were calculated from the $y_0(a_0)$ using the Riccati equation (Piaggio 1956, p. 201).

The Riccati equation can be obtained directly from the Schrödinger equation. Let $u_L(r)$ be a solution of the equation

$$\frac{d^2u_L(r)}{dr^2}-\left\{-k^2+V(r)+\frac{L(L+1)}{r^2}\right\}u_L(r) = 0 \tag{12}$$

where $V(r)$ is any potential, expressed in units of (length)$^{-2}$. The logarithmic derivative of $u_L(r)$

$$Y_L(r) = \frac{\{du_L(r)/dr\}}{u_L(r)}$$

satisfies the equation

$$\frac{dY_L(r)}{dr} = -k^2+V(r)+\frac{L(L+1)}{r^2}-Y_L{}^2(r). \tag{13}$$

Equation (13) is a special example of the Riccati equation

$$\frac{dY}{dr} = P(r)+Q(r)Y+R(r)Y^2.$$

In the present case $V(r)$ is either the Coulomb potential or the electric potential. Hence, for S-wave scattering,

$$\frac{dy_0(r)}{dr} = -k^2+\frac{b}{r}-y_0{}^2(r) \tag{14}$$

and

$$\frac{dy_0^{\mathrm{E}}(r)}{dr} = -k^2 + V_{\mathrm{E}}(r) - \{y_0^{\mathrm{E}}(r)\}^2 \tag{15}$$

where $y_0(r)$ and $y_0^{\mathrm{E}}(r)$ are defined by equations (1) and (10) for arbitrary r. At a large distance R we have, from equations (1), (8), (10) and (11), that

$$y_0(R) \simeq y_0^{\mathrm{E}}(R). \tag{16}$$

Hence, in principle, the problem is straightforward. From $y_0(a_0)$ we can calculate $y_0(R)$ using equation (14), and from $y_0^{\mathrm{E}}(R)$ $(= y_0(R))$ we can calculate $y_0^{\mathrm{E}}(a_0)$ by integrating equation (15). The coefficients $B_{0,i}^{\mathrm{E}}$ $(i = 0, 1, 2)$ are then obtained from a least-squares fit to these experimental values of $y_0^{\mathrm{E}}(a_0)$. The low-energy behaviour of $y_0^{\mathrm{E}}(a_0)$ can thus be interpolated and extrapolated. From a reversal of the above procedure, the logarithmic derivative $y_0(a_0)$ can also be extrapolated to lower energies.

In practice, the numerical integration of the Riccati equation is complicated by the presence of poles in $y_0(r)$, which necessitate the use of suitable transformations. The first transformation considered was

$$y_0(r) = k \cot\left\{\frac{f_1(r)}{2k} + kr\right\} \tag{17}$$

where $f_1(r)$ satisfies the differential equation

$$\frac{df_1(r)}{dr} = -\frac{2b}{r}\sin^2\left\{\frac{f_1(r)}{2k} + kr\right\}.$$

$f_1(r)$ is monotonic decreasing since

$$-\frac{2b}{r} \leqslant \frac{df_1}{dr} \leqslant 0.$$

For low-energy scattering a substantial part of the domain of integration for the Riccati equation arises from the region $r < r'$ where r' is the classical turning point, or transition line, i.e. $k^2 = V(r')$. As a result, a better transformation than (17) in this region would be one which involves the hyperbolic function rather than the circular one. The hyperbolic function, however, is not suitable for $r > r'$, on account of the poles of $y_0(r)$. The transformation

$$\left.\begin{matrix} y_0(r) \\ y_0^{\mathrm{E}}(r) \end{matrix}\right\} = \begin{cases} \{k^2 - V(r)\}^{1/2} \cot[\{k^2 - V(r)\}^{1/2} f_2(r)] & k^2 > V(r) \\ f_2^{-1}(r) & k^2 = V(r) \\ \{V(r) - k^2\}^{1/2} \cot[\{V(r) - k^2\}^{1/2} f_2(r)] & k^2 < V(r) \end{cases} \tag{18}$$

gives effect to these considerations and was found to be an improvement on (17). The potential term in equation (18) is Coulomb for $y_0(r)$ and electric for $y_0^{\mathrm{E}}(r)$. The function $f_2(r)$ is continuous across the transition line and obeys the differential equation

$$\frac{df_2(r)}{dr} = 1 + \frac{1}{2}\frac{dV(r)}{dr}\{k^2 - V(r)\}^{-1}\{f_2(r) - g(r)\} \tag{19}$$

where

$$g(r) = \frac{\sin[2\{k^2 - V(r)\}^{1/2} f_2(r)]}{2\{k^2 - V(r)\}^{1/2}} = \frac{\sinh[2\{V(r) - k^2\}^{1/2} f_2(r)]}{2\{V(r) - k^2\}^{1/2}}.$$

It should be noted that the transformation (18) is applicable only in regions r where

$$y_0^2(r) > V(r) - k^2 \tag{20}$$

on account of the branch cut of the function arccoth (z) from $-1 \leqslant z \leqslant 1$.

Before presenting the results of the exact calculations using these transformations, we shall consider further the Riccati equation. In particular, we shall investigate the low-energy

behaviour of $y_0(a_0)$. This function depends on δ_0, the combined nuclear *plus* vacuum polarization phase shifts in the presence of the Coulomb field (equation (11)). So far we have considered the effect of the nuclear potential embedded in the Coulomb and vacuum polarization potentials. We now consider the effect of the vacuum polarization potential embedded in the Coulomb and nuclear potentials. The final result, i.e. the energy behaviour of $y_0(a_0)$ is, of course, the same.

Let us consider the scattering by a short-range nuclear potential in the presence of the Coulomb field only. The effective-range formula is given by

$$\bar{y}_L(a_L) = k\left(\frac{G_L' + F_L' \cot \alpha_L}{G_L + F_L \cot \alpha_L}\right)_{a_L} = \bar{B}_{0,0} + \bar{B}_{0,1}k^2 + \bar{B}_{0,2}k^4 \tag{21}$$

where α_L is the nuclear phase shift appropriate to this particular situation. From this value of $\bar{y}_L(a_L)$, we wish to calculate the value of $y_L(a_L)$ arising from the vacuum polarization. Writing

$$\bar{y}_L(r) = k\left\{\frac{(G_L' \sin \alpha_L + F_L' \cos \alpha_L) - x_L(r)(G_L' \cos \alpha_L - F_L' \sin \alpha_L)}{(G_L \sin \alpha_L + F_L \cos \alpha_L) - x_L(r)(G_L \cos \alpha_L - F_L \sin \alpha_L)}\right\} \tag{22}$$

with the boundary condition $x_L(a_L) = 0$, we find from the Riccati equation that

$$\frac{dx_L(r)}{dr} = \frac{V_{\mathrm{vp}}(r)}{k}\{(G_L \sin \alpha_L + F_L \cos \alpha_L) - x_L(r)(G_L \cos \alpha_L - F_L \sin \alpha_L)\}^2 \tag{23}$$

where $V_{\mathrm{vp}}(r) = V_{\mathrm{E}}(r) - b/r$ is the vacuum polarization potential. The experimental phase shift is given by

$$\delta_L = \alpha_L - \arctan\{x_L(\infty)\}. \tag{24}$$

We note that α_L is not equal to $\delta_L{}^{\mathrm{E}}$. From equations (1) and (24) we have

$$\delta y_L(a_L) \equiv y_L(a_L) - \bar{y}_L(a_L)$$
$$= \frac{kx_L(\infty)}{[(G_L \sin \alpha_L + F_L \cos \alpha_L)\{(G_L \sin \alpha_L + F_L \cos \alpha_L) - x_L(\infty)(G_L \cos \alpha_L - F_L \sin \alpha_L)\}]_{a_L}}. \tag{25}$$

From equations (21), (23) and (25) we could calculate the function $\delta y_L(a_L)$ exactly provided the coefficients $\bar{B}_{0,i}$ $(i = 0, 1, 2)$ were known. A good approximation to these coefficients can be obtained by using the numbers given in table 1. For the alpha case small adjustments in these numbers are necessary in order to produce the *observed* resonance at the correct energy. This is similar to the approach by Foldy and Eriksen. In practice, there is the difficulty of the numerical calculation of the irregular Coulomb wave function over a large range of r. This difficulty is avoided by the method of § 5.

A more drastic approximation to equation (25) can be made if the vacuum polarization potential is treated as a perturbation. The perturbation theory result is obtained by neglecting the term $x_L(r)$ on the right-hand side of equation (23) and the term $x_L(\infty)$ in the denominator of equation (25) to give

$$\delta y_L(a_L) \simeq \int_{a_L}^{\infty} V_{\mathrm{vp}}(r) \frac{(G_L + F_L \cot \alpha_L)^2}{(G_L + F_L \cot \alpha_L)^2_{r=a_L}}. \tag{26}$$

Equation (26) was used by Foldy and Eriksen in their analysis of the proton data. Since the vacuum polarization potential is repulsive, $\delta y_L(a_L)$ is positive.

In this perturbation approach the approximation

$$|(G_L \sin \alpha_L + F_L \cos \alpha_L)_{a_L}| \gg |x_L(\infty)(G_L \cos \alpha_L - F_L \sin \alpha_L)_{a_L}|$$

is made. At low energies, however, the vacuum polarization potential provides a larger contribution to the phase shift δ_L than the nuclear potential, and at very low energies we

can make the opposite approximation

$$|(G_L \sin\alpha_L + F_L \cos\alpha_L)_{a_L}| \ll |x_L(\infty)(G_L \cos\alpha_L - F_L \sin\alpha_L)_{a_L}|. \tag{27}$$

Equation (25) then becomes

$$\begin{aligned}\delta y_L(a_L) &\simeq -\frac{k}{\{(G_L \sin\alpha_L + F_L \cos\alpha_L)(G_L \cos\alpha_L - F_L \sin\alpha_L)\}_{a_L}} \\ &\simeq -\frac{k\cot\alpha_L}{\{G_L(G_L + F_L \cot\alpha_L)\}_{a_L}} \\ &= k\left(\frac{G_L'}{G_L}\right)_{a_L} - k\left(\frac{G_L' + F_L' \cot\alpha_L}{G_L + F_L \cot\alpha_L}\right)_{a_L}\end{aligned}$$

so that

$$y_L(a_L) \simeq k\left(\frac{G_L'}{G_L}\right)_{a_L} = z_L(a_L). \tag{28}$$

The ranges of validity of these two approximations can be judged from the exact calculations presented in the next section.

5. Effective-range analysis (ii): vacuum polarization and short-range forces

5.1. *p–p scattering*

From each of the 34 values of $y_0(0{\cdot}48)$ used in the previous analysis (§ 2), $f_1(0{\cdot}48)$ was calculated using the inverse of equation (17). The differential equation for $f_1(r)$ was integrated numerically, by the Runge–Kutta method, on the English Electric KDF9 computer at Liverpool University. At a large distance R equation (16) was used to give

$$f_1^{\mathrm{E}}(R) = f_1(R).$$

The integration was then performed in the reverse direction with the Coulomb potential replaced by the electric potential. For an accuracy of five figures in the value of $y_0^{\mathrm{E}}(0{\cdot}48)$ it was found that R could be taken as 400 fm ($z \simeq 2$), except for a few low-energy points which required a greater matching distance. From a least-squares fit to $y_0^{\mathrm{E}}(0{\cdot}48)$ the coefficients given in table 2 were obtained.

Table 2. Coefficients $B_{0,i}^{\mathrm{E}}$ obtained from the experimental p–p phase shifts

	No. of points	$B_{0,0}^{\mathrm{E}}$ (fm^{-1})	$B_{0,1}^{\mathrm{E}}$ (fm)	$B_{0,2}^{\mathrm{E}}$ (fm^3)	S^2
p–p (1)	29	0·025208	0·93893	−1·6868	1·79
p–p (2)	5	0·024997	0·92630	−1·1055	0·35

$a_0 = 0{\cdot}48$ fm.

If we compare the values of S^2 in tables 1 and 2, it is seen that the analyses which included vacuum polarization gave a slightly improved fit. Furthermore, the coefficient $B_{0,2}^{\mathrm{E}}$ for the second set of data is negative, and is therefore of opposite sign to $B_{0,2}$ (table 1). A similar result for the case of no hard core would mean that the shape parameter is *positive*. However, the calculation of $B_{0,2}^{\mathrm{E}}$, for $a_0 = 0$, is difficult and will not be attempted here. Instead we shall estimate the coefficients $\bar{B}_{0,i}$ ($a_0 = 0$) and compare the results with those of Noyes (1964).

We assume that since the vacuum polarization potential is much smaller than the Coulomb potential it is a good approximation to take $\bar{B}_{0,i} = B_{0,i}^{\mathrm{E}}$ ($i = 0, 1, 2$) for the hard core radius of 0·48 fm. From the numbers given in table 2, the scattering length a, the effective range r_e and the shape parameter P were calculated. The results, together with those of Noyes, are given in table 3. The labels attached to the analyses by Noyes are explained in his paper (Noyes 1964).

Table 3

	$B_{0,2}(a_0 = 0)$	a (fm)	r_e (fm)	P	ϵ
BC†	0·6984	−7·8009	2·687	−0·036	6·88
SI†	0	−7·8163	2·746	0	1·82
p–p (2)‡	−0·4933	−7·8342	2·785	0·023	0·88
PWDR†	−0·5190	−7·8259	2·786	0·024	0·87
SD†	−0·5671	−7·8284	2·794	0·026	0·86
p–p (1)‡	−1·0948	−7·8215	2·812	0·049	2·0
CFS†	−1·4212	−7·8426	2·853	0·0612	5·67

† All these abbreviations are defined by Noyes (1964).

‡ These numbers were obtained by taking $\bar{B}_{0,i} = B_{0,i}{}^{E}$ $(i = 0, 1, 2)$ for $a_0 = 0{\cdot}48$ fm.

To compare the results we define the quantity ϵ by $(n-m)\epsilon = \chi^2 = nS^2$, where S^2 is given by equation (4) and m is the number of free parameters used to fit the experimental data. The best fit is obtained when ϵ is least. We see that the analysis p–p (2) gave a fit to the data comparable with the best fit obtained by Noyes. The p–p (2) analysis did not differ greatly from the SD analysis of Noyes. From table 3 it is also seen that the more the shape parameter differed from the value 0·026 the worse the fit became. It is interesting to note that the hard core effective-range fit to the p–p (1) data gave a better value for P than the BC or CFS models of Noyes.

The boundary condition (BC) used by Noyes was not the same as the boundary condition at the hard core used in the present analysis. For S-wave scattering in the absence of the Coulomb and vacuum polarization forces, it is possible to choose a distance $\bar{r}$ for which the expansion of the logarithmic derivative, $k \cot(\delta_0 + k\bar{r})$, does not contain a term in k^2. Neglecting terms of order k^4, we have $k \cot(\delta_0 + k\bar{r}) = B$, where B is a constant. This equation can be written

$$k \cot \delta_0 = \frac{k(B + k \tan k\bar{r})}{k - B \tan k\bar{r}} = B'(k)$$

say. For his BC model Noyes assumed that a similar energy behaviour held for $y_0(0)$ (equation (3)), i.e. $y_0(0) = B'(k)$. It is then difficult to associate the formula with the energy dependence of the logarithmic derivative. A slightly better BC fit would probably be obtained by considering $y_0(\bar{r}) = B$.

The coefficients for p–p (1) given in table 2 were used to calculate $y_0{}^{E}(0{\cdot}48)$ and hence $y_0(0{\cdot}48)$. The results are shown in figures 2 and 3. Above 10 kev the behaviour is similar

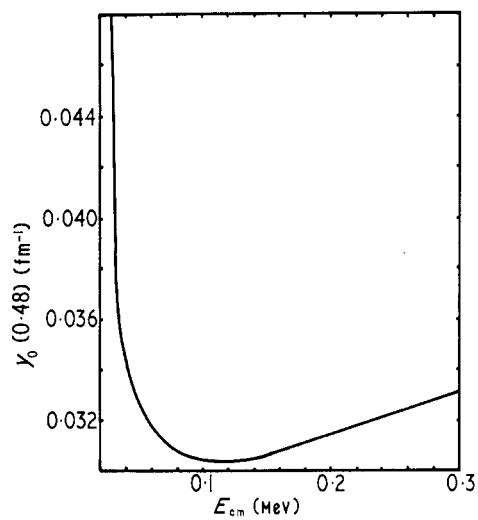

Figure 2. Proton–proton scattering. Energy dependence of $y_0(0{\cdot}48)$ resulting from $y_0{}^{E}(0{\cdot}48) = 0{\cdot}025\,208 + 0{\cdot}938\,93\,k^2 - 1{\cdot}6868\,k^4$.

to that found by Foldy and Eriksen, i.e. the deviation from a quadratic behaviour of the function $y_0(a_0)$ due to vacuum polarization is such that $y_0(a_0)$ increases at low energies. However, $y_0(0{\cdot}48)$ becomes infinite at about 9 kev, and at lower energies is asymptotic to $z_0(0{\cdot}48)$, as predicted by equation (28). Perturbation theory, therefore, is definitely

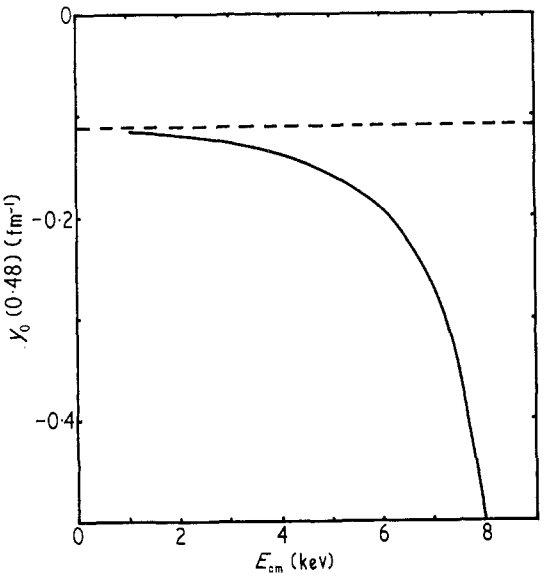

Figure 3. Proton–proton scattering. Energy dependence of $y_0(0{\cdot}48)$ resulting from $y_0^{\mathrm{E}}(0{\cdot}48) = 0{\cdot}025\,208 + 0{\cdot}938\,93\,k^2 - 1{\cdot}6868\,k^4$.
The broken curve is the function $z_0(0{\cdot}48)$.

unsuitable below about 10 kev. The singular point of $y_0(0{\cdot}48)$ arises because the phase shift due to the nuclear attraction is exactly cancelled by that due to vacuum polarization. At this energy δ_0 is simply the hard core phase shift, i.e. $\tan \delta_0 = -(F_0/G_0)_{0.48}$. For energies below about 10 kev the experimental phase shift δ_0 is predicted to be *negative*.

5.2. *α–α scattering*

For the calculation of $y_0^{\mathrm{E}}(1{\cdot}7)$ from $y_0(1{\cdot}7)$ the integration of the Riccati equation was performed using the second transformation (equation (18)). It was found that this transformation could not be applied directly to $y_0(1{\cdot}7)$ because the inequality (20) was not satisfied. The calculations were performed in two stages. Equation (14) was integrated from 1·7 fm to r_0, where r_0 was the smallest distance for which

$$y_0^2(r_0) > 1{\cdot}01\left(\frac{b}{r_0} - k^2\right).$$

The inverse of the transformation (18) was used to calculate $f_2(r_0)$ and equation (19) was integrated from r_0 to the large distance R. From the value of $f_2(R)$, equation (19), with the Coulomb potential replaced by the electric potential, was integrated until the condition $\{V_{\mathrm{E}}(r_0') - k^2\}^{1/2} f_2(r_0') \geqslant 2{\cdot}65 \simeq \operatorname{arccoth}(1{\cdot}01)$ was satisfied. Using the value of $y_0^{\mathrm{E}}(r_0')$ calculated from equation (18), the value of $y_0^{\mathrm{E}}(1{\cdot}7)$ was obtained from the integration of equation (15). This procedure was carried out for each of the 37 experimental points, together with their errors, using the IBM 7040 computer at Liverpool University. On account of the large range of integration, it was necessary to use the double precision number facility of FORTRAN IV. The coefficients $B_{0,i}^{\mathrm{E}}$ $(i = 0, 1, 2)$ obtained from the least-squares fit to the $y_0^{\mathrm{E}}(1{\cdot}7)$ are given in table 4.

Table 4. Coefficients $B_{0,i}^{E}$ obtained from the 37 experimental α–α phase shifts

E_r (kev)	a_0 (fm)	$B_{0,0}^{E}$ (fm^{-1})	$B_{0,1}^{E}$ (fm)	$B_{0,2}^{E}$ (fm^3)	S^2
92·12	1·7	−0·45735	0·78750	0·01559	0·45
94·5	1·7	−0·45719	0·78640	0·01657	0·44

Comparing the values of S^2 in tables 1 and 4, we see that for the alpha case the fit is slightly worse when vacuum polarization is included. This discrepancy is small and is probably due to the comparatively large errors in the experimental phase shifts. It would be useful if the α–α phase shifts could be determined to the same accuracy as the p–p phase shifts.

The low-energy behaviour of $y_0(1{\cdot}7)$ resulting from these coefficients (E_r = 94·5 kev) is shown in figures 4 and 5. We see that the perturbation approach is unsuitable for energies below about 94 kev. As the energy decreases, $y_0(1{\cdot}7)$ approaches the function $z_0(1{\cdot}7)$.

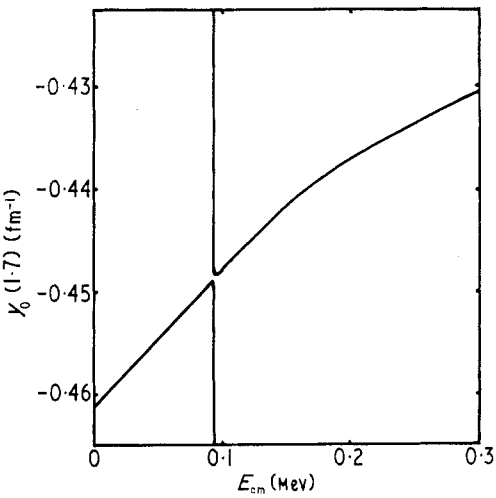

Figure 4. α–α scattering. E_r = 94·5 kev. Energy dependence of y_0 (1·7) resulting from $y_0^{E}(1{\cdot}7) = -0{\cdot}457\,19 + 0{\cdot}786\,40\,k^2 + 0{\cdot}016\,57\,k^4$.

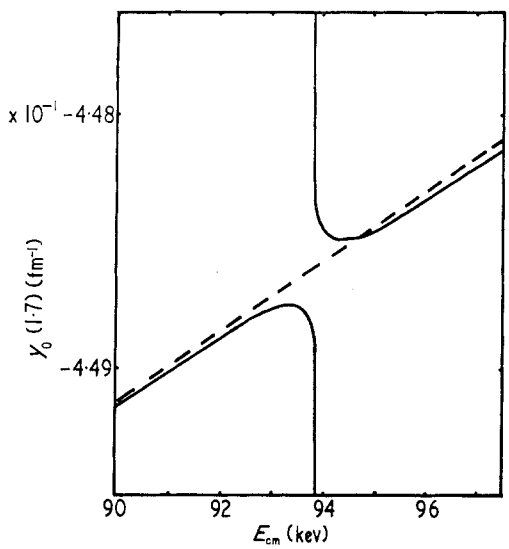

Figure 5. α–α scattering. E_r = 94·5 kev. Energy dependence of $y_0(1{\cdot}7)$ resulting from $y_0^{E}(1{\cdot}7) = -0{\cdot}457\,19 + 0{\cdot}786\,40\,k^2 + 0{\cdot}016\,57\,k^4$.
The broken curve is the function $z_0(1{\cdot}7)$.

From figure 4 it was found that, if the phase shift δ_0 for α–α scattering at energies in the region 0·1–0·3 MeV (c.m.) could be measured to an accuracy of 0·2°, we should have direct evidence for the vacuum polarization effect in scattering. The experimental results at 0·2 MeV and 0·3 MeV have errors of 0·5° and 1·0° respectively.

In the region of the resonance, $y_0(1{\cdot}7)$ was found to have the energy dependence

$$y_0(1{\cdot}7) = A + BE + \frac{C}{E-\bar{E}}. \tag{29}$$

The width of the ground state of beryllium was calculated from equations (5) and (29). It should be remembered that the value of $(y_0(1{\cdot}7))_{E_r}$ was fixed. This proved to be a powerful constraint. Initially, $(y_0(1{\cdot}7))_{E_r}$ was determined to an accuracy of five figures. However, as a result of the pole in $y_0(1{\cdot}7)$, it was found necessary to work to an accuracy of seven figures for the calculation of $y_0(1{\cdot}7)$ in the region of the resonance. This accuracy was not precluded by the experimental accuracy. The experimental errors gave rise to uncertainties in the coefficients $B_{0,1}{}^{\mathrm{E}}$ and $B_{0,2}{}^{\mathrm{E}}$. For *fixed* values of $B_{0,1}{}^{\mathrm{E}}$ and $B_{0,2}{}^{\mathrm{E}}$ within the experimental error, the value of $B_{0,0}{}^{\mathrm{E}}$ was calculated to give $(y_0{}^{\mathrm{E}}(1{\cdot}7))_{E_r}$ to an accuracy of seven figures. For example, the value of $B_{0,0}{}^{\mathrm{E}}$ in table 4 became $-0{\cdot}457\,346\,0$ for $E_r = 92{\cdot}12$ kev. The effect of the errors in the coefficients $B_{0,i}{}^{\mathrm{E}}$ on the width was small compared with that arising from the error in the location of the resonance. Separate calculations were performed for various values of E_r and the results were used to determine the parameters A, B, C and $\bar{E}$ (equation (29)). These are given in table 5.

Table 5

E_r (kev)	A (fm^{-1})	B (fm^{-1} MeV^{-1})	$C\times10^7$ (fm^{-1} MeV)	$\bar{E}$ (kev)	Γ (ev)
92·07	−0·46129	0·13484	0·1468	91·570	6·11
92·12	−0·46129	0·13484	0·1480	91·618	6·14
92·17	−0·46129	0·13486	0·1493	91·665	6·18
94·5 (7 figs)	−0·46128	0·13479	0·2212	93·887	7·7
94·5 (5 figs)	−0·46128	0·13474	0·2204	93·869	8·1

We see from table 5 that the value of Γ is the same (to two decimal places) as that found from the previous analysis (§ 2). In fact, the decrease in the value of Γ due to the inclusion of the vacuum polarization effect was approximately 0·1%. The results of the five-figure analysis are also given in table 5.

5.3. *Computational difficulties*

Certain computational difficulties were encountered when the calculations were extended to very low energies. It turned out that these difficulties were not important for any of the results given in this paper. If, however, the resonance energy for α–α scattering had been 10 kev, say, the calculation of Γ would have been impossible by the numerical techniques used here. In table 6 the results of the numerical calculations for α–α scattering at an energy of 10 kev are given.

Table 6

	$y_0(1{\cdot}7)$	$y_0{}^{\mathrm{E}}(1{\cdot}7)$	$y_0(401{\cdot}7)$	$y_0{}^{\mathrm{E}}(401{\cdot}7)$
(1)		−0·454, −0·465	E →	0·02225085
(2)		−0·46136	← E	0·02225085
(3)	−0·45984	← C		0·02225085
(4)	−0·465	C →	0·02224907	
(5)	−0·45984	← C	0·02224907	

The initial and final values of the logarithmic derivative lie at the tail and head of the arrow respectively, for example the integration of the Riccati equation for the Coulomb potential (C) with the boundary condition $y_0(r) = -0{\cdot}465$ at $r = 1{\cdot}7$ fm gave the result $y_0(r) = 0{\cdot}022\,249\,07$ at $r = 401{\cdot}7$ fm.

The discrepancies in these results arise from the enormity of the irregular wave functions compared with the regular ones. Let us consider the fourth example in table 6. This involves the Coulomb potential only. The initial value of y_0 (1·7), i.e. $-0{\cdot}465$, differs from the value of $z_0(1{\cdot}7)$ by 1%, so that cot δ_0 is very large. Hence at a distance of 401·7 fm, where the regular and irregular Coulomb wave functions are comparable in magnitude, $y_0(401{\cdot}7)$ is equal to $(kF_0'/F_0)_{401{\cdot}7}$ plus a very small correction. The rounding-off error in the value of $y_0(401{\cdot}7)$ is such that the value of cot δ_0, although still very large, is reduced by several orders of magnitude, and consequently the value of $y_0(1{\cdot}7)$ resulting from the inward integration is equal to $z_0(1{\cdot}7)$ plus a very small correction. The value of $z_0(1{\cdot}7)$ obtained directly from the Coulomb wave functions (§ 2) was $-0{\cdot}459\,84$. From a similar argument for the integration involving the electric potential, we can conclude that the value of $(kT_0'/T_0)_{1{\cdot}7}$ at 10 kev is $-0{\cdot}461\,36$.

5.4. *The radius of the hard core*

The results given in this paper are those for a fixed value of the hard core radius, i.e. 0·48 fm for the proton case and 1·7 fm for the alpha case. The effect of small changes in these values was investigated. It was found that a small variation in a_0 could be compensated by small changes in the coefficients $B_{0,i}{}^{\mathrm{E}}$ (cf. Kermode 1967). Also the energy dependence of $y_0(a_0)$ was found to be similar to that shown in figures 2–5, except that the singularity occurred at a higher energy for a smaller hard core radius (and vice versa).

The value of Γ calculated from the α–α phase shifts is, of course, independent of the hard core radius chosen. Equation (5) can be written $\Gamma = 2(d\delta_0/dE)_{E_r}{}^{-1}$. However, for a hard core radius which differs from 1·7 fm by more than about 0·5 fm, the effect of the coefficient $B_{0,3}{}^{\mathrm{E}}$ must be considered. This point is discussed by Kermode (1967) for the case of no hard core and no vacuum polarization.

6. Conclusion

Exact calculations have shown that the effects of vacuum polarization on S-wave p–p scattering are of the same order of magnitude as the experimental errors. The effective-range analysis for the proton case gives a slightly better fit when the effect is explicitly included. At low energies the phase shift due to the long-range vacuum polarization potential is greater than that due to the short-range nuclear interaction. Hence the experimental 'nuclear' phase shift δ_0 is *negative* at low energies.

For the alpha case the energy dependence of $y_0(1{\cdot}7)$ (equation (29)) differs considerably from that given by the *short-range* assumption (equation (2)). Nevertheless, the latter method gave the correct value of Γ to within 0·1%. There are two reasons for this. First, the error in each experimentally determined phase shift δ_0 is much larger than the vacuum polarization contribution. More accurate data would show that equation (2) is not valid. Secondly, the resonance energy E_r is an important factor in the calculations. For a given resonance energy the width is related to barrier penetration. In § 2 the barrier was assumed to be pure Coulomb. In § 5.2 the calculations also took into account the additional vacuum polarization barrier. The vacuum polarization potential has the value 5·8 kev (0·5% of Coulomb potential) at 4·6 fm and the value 0·11 kev (0·1% of Coulomb potential) at the classical turning point. Hence the percentage increase in the height of the barrier due to vacuum polarization is approximately the same as the percentage decrease in the calculated value of Γ.

If the vacuum polarization potential could be 'switched-off', both the resonance energy and the corresponding width would decrease. A decrease of 1 kev in E_r would lead to a decrease of 0·6 ev in Γ. It would be an interesting academic exercise to calculate the resonance energy for this situation.

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Appendix. Evaluation of the Schwinger integral

Writing $2\kappa r = z$, we have

$$I(z) = \int_1^\infty dx\, \mathrm{e}^{-zx}\left\{\left(\frac{1}{x^2}+\frac{1}{2x^4}\right)F_1(x) - F_2(x)\right\}$$

where

$$F_1(x) = x-(2x)^{-1}-(8x^3)^{-1}-(16x^5)^{-1}-5(128x^7)^{-1}-7(256x^9)^{-1}$$

and

$$F_2(x) = \{F_1(x)-(x^2-1)^{1/2}\}\left(\frac{1}{x^2}+\frac{1}{2x^4}\right).$$

The integrals involving $F_1(x)$ may be written in terms of exponential integrals, i.e.

$$I_1(z) = E_1(z)-\tfrac{3}{8}E_5(z)-\tfrac{1}{8}E_7(z)-\tfrac{9}{128}E_9(z)-\tfrac{3}{64}E_{11}(z)-\tfrac{7}{512}E_{13}(z).$$

$E_1(z)$ was calculated from the polynomial approximation ($0 \leqslant z \leqslant 1$) or the rational approximation ($z > 1$) (Abramowitz and Stegun 1965, p. 231). The remaining exponential integrals were calculated from $E_1(z)$ by using the recurrence relations

$$E_{n+1}(z) = \{\exp(-z)-zE_n(z)\}\frac{1}{n} \qquad (z \leqslant 5).$$

The function $F_2(x)$ is very small for $x \geqslant 2$. In the region $1 \leqslant x \leqslant 2$, $F_2(x)$ was approximated by the function

$$\sum_{i=1}^{n} a_i \exp\{-b_i(x-1)\}.$$

The coefficient b_i was calculated from the root $y_i = \exp(-b_i h)$ of the equation

$$\begin{vmatrix} 1 & y & y^2 & . & y^n \\ d_1 & d_2 & d_3 & . & d_{n+1} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_n & d_{n+1} & d_{n+2} & . & d_{2n} \end{vmatrix} = 0$$

where $d_r = F_2\{1+(r-1)h\}$, $r = 1, \ldots, 2n$, and h was a chosen step length. The coefficients a_i ($i = 1, \ldots, n$) were calculated from the simultaneous equations

$$\sum_{i=1}^{n} a_i y_i^{j-1} = d_j.$$

If we take $n = 5$, the coefficients

$$\begin{aligned} a_1 &= 0{\cdot}000\,004 & b_1 &= 0 \\ a_2 &= 0{\cdot}064\,663 & b_2 &= 12{\cdot}4374 \\ a_3 &= 0{\cdot}115\,513 & b_3 &= 26{\cdot}7971 \\ a_4 &= 0{\cdot}072\,421 & b_4 &= 65{\cdot}9294 \\ a_5 &= 0{\cdot}116\,540 & b_5 &= 221{\cdot}331 \end{aligned}$$

provided a good fit to $F_2(x)$. These coefficients were used to calculate

$$I_2(z) = \sum_{i=1}^{5} \frac{a_i \, e^{-z}}{z+b_i} [1 - \exp\{-(z+b_i)\}].$$

Using $dE_n(z)/dz = -E_{n-1}(z)$ and $E_0(z) = z^{-1} \exp(-z)$, the calculation of $dI(r)/dr$ was straightforward.

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